

## Two faces of vectors

### Student Materials

*This article makes connections between the vectors you've drawn as arrows and concepts and techniques you've learned in math classes. You can use these connections to math to make scientific results more accurate and faster and to automate them through computer operations.*

### Vectors in Science vs. Math Classes

As Earth's crust moves, GPS stations anchored to it with concrete also move. Recording a station's position through time allows us to visualize its movement as a vector. To understand vectors and how they differ from ordinary numbers, view the Khan Academy's video, ["Introduction to Vectors and Scalars."](#) (8:39. Search for "Khan vectors scalars.") You might have already examined GPS N - S and E - W velocity vectors and added them graphically in "Velocity maps."

### Vectors — Finding Length with the Pythagorean Theorem

You can measure how long the arrow denoting a vector is and use a scale to learn its absolute length. However, you can also calculate the length of the total horizontal velocity vector from N - S and E - W velocity vectors with the Pythagorean Theorem. Examples in "Velocity maps" work out how to do this. The Khan Academy's video ["Pythagorean Theorem"](#) explains it and works a few examples. (10:46. Search for "Khan Pythagorean.")

### Vectors, sines, and cosines

You can use a protractor or compass to measure the angle a vector makes with north—the "azimuth." Mathematics, however, makes it possible to calculate the azimuth without drawings. With right triangles, vectors-as-arrows handily connect with sines and cosines from Algebra 1, Geometry, or Trigonometry. Remember that the sine is the side opposite the angle  $\theta$  divided by the hypotenuse. The cosine is the side adjacent to angle  $\theta$  divided by the hypotenuse. If you're rusty about this, the Khan Academy video ["Basic Trigonometry"](#) will help. (9:17. Search for "Khan trigonometry.")



GPS data gives you the lengths of the opposite and the adjacent sides, and you can calculate the hypotenuse with the Pythagorean theorem. The first two are the velocity vectors from time-series graphs, and the hypotenuse is the total horizontal velocity vector. Thus, you can find the azimuth of the total velocity vector by calculating the sine or cosine, the inverse of which tells you the angle. (Alternatively, you could find the inverse of the tangent.)

If you could use a protractor, why would you find the azimuth this way? You can get the same information two different ways and check their results. Also, computers use sines or cosines, not a protractor; they automate and speed up the calculation vector's azimuth. This is the kind of calculation that the spreadsheet-based strain calculator provides. Finally, we can do better than shifting arrows around, drawing, and measuring: A value calculated analytically is more likely to be accurate than one determined graphically. The next section expands on calculating the total horizontal vector.

## Adding Vectors Analytically

You could think of a N - S velocity vector from a GPS station lying along a y-axis and an E - W velocity vector along an x-axis. As before, you make a total horizontal vector by adding the two legs. With a clever way to describe the N - S and E - W vectors, you can add them analytically—you do not need to draw them. “Unit vector notation” provides that ability. The Khan Academy video ["Unit Vector Notation"](#) shows these concepts with examples. (9:54. Search for “Khan unit notation.”) The video shows the reverse of what you will do, however: Mr. Khan decomposes a vector into two vectors that lie along the x- and y-axes, and he articulates the concepts using formal mathematics.

## Vectors as (x, y) Ordered Pairs

Long ago you learned to graph ordered pairs on x- and y- axes of a Cartesian coordinate system. Even before that, you may have learned to locate a point on a grid by playing Battleship or chess. You know that an ordered pair is a point on a graph.

You can also represent a vector as an ordered pair. The simplest case is a vector that begins at the origin (0, 0) and ends at the ordered pair (x, y). We use the ordered pair to mark the endpoint of the vector. However, vectors don't need to start at the origin; they could start at any pair of coordinates. Figure 1 shows the same vector with several different origins—but with the same length and orientation. In these cases the head of the vector has the coordinates  $(3 + x_{\text{origin}}, 2 + y_{\text{origin}})$ .

## Vectors as matrices

The next step is to think of an ordered pair as a *sequence of numbers* that represents a point or the head of a vector. That sequence of numbers can be represented as a simple matrix. As a matrix, the sequence *looks* different than an ordered pair, and you can do powerful things with it very quickly, especially if you're using a graphing calculator or computer.

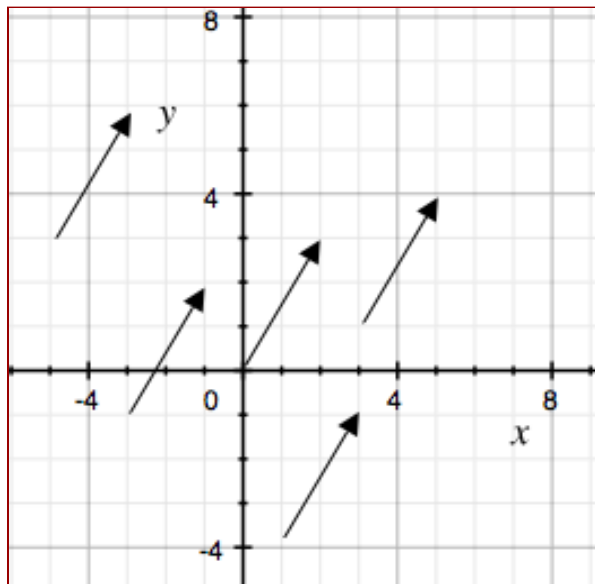


Figure 1. Identical vectors. The origin of one lies at the origin (0, 0). Its head lies at (2, 3). The others have the same direction and magnitude but different starting points.

More generally, a matrix is an array of numbers upon which you can do math. Why use matrices? First, you can manipulate them easily and quickly. Also, they are used in the spreadsheet-based strain calculator in this activity about deformation of Earth's crust. While you can analyze much of the GPS data by hand and visually, to get precise results, you must use the strain calculator. It uses matrices.

Let's start with how a matrix looks. Matrices are often shown in brackets. For example the ordered pair (2, 4) in Figure 2 could be represented as the matrix **a**, where

$\mathbf{a} = \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} 2 & 4 \end{bmatrix}$  or  $\mathbf{a} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ . We'll use the column matrix here, and we'll call it what mathematicians call a one column or one row matrix—a “vector.” (No accident, no coincidence.) Another way to think of this vector **a** is graphically. (Fig. 2.)

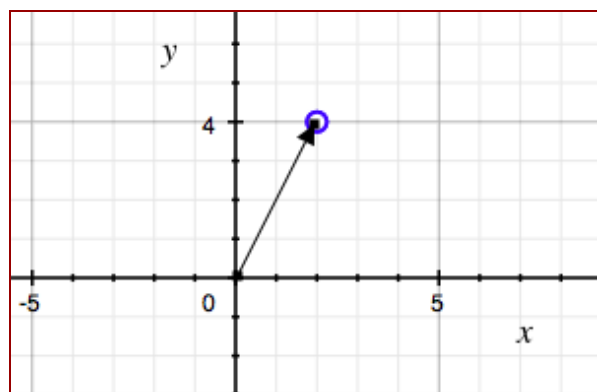


Figure 2. We could describe the point shown in the circle as having the coordinates (2, 4) or as a vector originating at the origin. They could be represented by matrix  $\mathbf{a}$ , where  $\mathbf{a} = \begin{bmatrix} 2 & 4 \end{bmatrix}$ .

More complex matrices look much like a table or data in a spreadsheet. For example, a four row, three column matrix is called a  $4 \times 3$  matrix and looks like this:  $\mathbf{B} = \begin{bmatrix} 3 & -7 & 12 & 2 & 15 & -1 & 0 & 1 & 0 & -82 & 6 & 5 \end{bmatrix}$ . The strain calculator uses  $6 \times 6$  matrices. You put the location of GPS stations and their velocity vectors into the strain calculator for it to

manipulate the data as matrices. The next section uses a series of resources to build on the idea of a vector as either an arrow or a matrix so that you can understand many of the concepts and operations done within the strain calculator. Because the Khan Academy videos were made for Physics or Linear Algebra, there will be a few small gaps to jump.

### Operation of Matrices and Vectors

The following table lists video tutorials about the basics of matrix math. They explain the material in the article, “A summary of vectors and vector arithmetic” and often provide examples.

Title	Notes	Details
<a href="#">Introduction to the matrix</a>	Explains matrix basics.	4:29. Search for “Khan introduction matrix 2013.”
<a href="#">Matrix addition and subtraction</a>	Teaches to add and subtract matrices.	5:35. Search for “Khan matrix addition subtraction video.”
<a href="#">Vector examples</a>	See how vectors-as-arrows in science and vectors-as-matrices in math merge in a sweet video of examples about adding vectors both ways.  The first two minutes are formal terminology; be patient and watch the first 11 minutes about vector addition--and then stop watching for the time being.	25:33. Search for “Khan vector examples.”
<a href="#">Scalar multiplication</a>	How to multiply a matrix by a scalar—e.g. how to double or triple the values in a matrix.	2:18. Search for “Khan scalar multiplication basic.”
<a href="#">Matrix multiplication (part 1)</a>	The mechanics of multiplying one matrix by another.	13:40. Search for “Khan matrix multiplication.”
<a href="#">Vector examples</a>	The rest of the video works through multiplication of a matrix by a scalar.	25:33. Search for “Khan vector examples.”

The Linear Algebra beyond those concepts is applied in the spreadsheet-based strain calculator. Among them, dot and cross products are commonly used. You will encounter them in the article, “A summary of matrices and matrix math,” in introductory college Physics, and in Linear Algebra.

Title	Notes	Details
<a href="#">Vector Dot Product and Vector Length</a>	Matrix based. Reviews addition of vectors and scalar multiplication. Done analytically: $a_1b_1 + a_2b_2 + \dots a_nb_n$ .  Then defines length of vector as the square root of the sum of squared vectors. Draws connection to Pythagorean theorem, but expands to multiple dimensions.	9:10. Search for “Khan vector length.”
<a href="#">The Dot Product</a>	Graphically based. Uses cosine. Example with work.	10:33. Search for “Khan dot product.”
<a href="#">Cross Product Introduction</a>	Matrix based. Sketches in a vector to show the 3 <sup>rd</sup> dimension determined by the right-hand rule.	15:47. Search for “Cross product introduction.”
<a href="#">Cross Product 1</a>	Graphically based. Uses sine. Works generic example and draws in a right-hand to show how the rule works.	10:02. Search for “Khan cross 1.”
<a href="#">Cross Product 2</a>	More examples. Same problem but from the opposite direction. Gives intuition about how the definition works.	10:52. Search for “Khan cross 2” and choose the second link.
<a href="#">Dot vs. Cross Product</a>	Graphical and analytical. Compares and contrasts dot and cross products and then makes sense of them graphically. Good review, plus an explanation for intuition.	10:45. Search for “Khan dot vs. cross.”
<a href="#">Calculating dot and cross products with unit vector notation</a>	Uses unit vector notation and ties explicitly to matrix bracket notation. Solves dot and cross products analytically. Explicitly uses matrices to solve for the cross product. Starts with graphical and trigonometry- based definition and, at the end, says that solutions work out graphically to be the same.  Overlaps “Cross Product Introduction” but also includes dot product with unit vector notation. A good summary.	9:46 Search for “Khan calculating dot and cross.”

For more advanced concepts or methods used in the strain calculator (e.g. transformations, transposes, inverses, determinants, and Eigenvectors and Eigenvalues), you will need to dig more into Linear Algebra. Two sections of the Khan Academy, “Matrix transformations” and “Alternative coordinate systems (bases),” *Linear Algebra for Dummies*, or a Linear Algebra textbook could be helpful resources.